Math 20D Spring 2023

WEEK · I.

§1. Some Logistics.

Online resources: Canvas, Gradescope, MATLAB Gradescope.

- For questions related to MATLAB please contact MATLAB instructors (info available in Canvas).
- Course Notes \ Recordings will be uploaded to Canvas.

Announcements:
- Homework 1 will be released tomorrow.
- office hours begin next week but discussion sections will start this week.
- Syllabus available in Canvas.
- Sections C03 & C04 will run in a remote format all quarter.

---

§2. Motivation for studying DE's.

Differential equations frequently arise when scientific methods are applied to studying the behavior over time of real world systems.

The differential equation serves as a mathematical description of how the system evolves from one moment to the next.

Analyze / Solve the D.E. (More mathematical step. Often exact solution cannot be found)

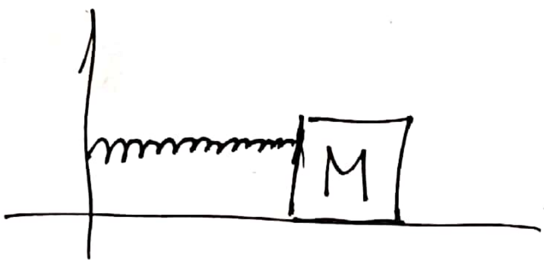
Real world system $\longrightarrow$ Differential equation (short term information) $\longrightarrow$ Prediction about long term behaviour of system

Mathematical Model (usually based on science)

Example: (Harmonic Oscillator).

A mass M is attached to a wall via a spring. The mass is placed on a table, the spring is extended and then released.

(Prototype of oscillatory motion)



Say the spring is released at time $t = 0$ and write $y(t)$ for the displacement of M relative to the springs equilibrium at time $t \geq 0$.

<u>Goal</u>: Predict the behaviour of M, or equivalently describe $y(t)$ as a function of t.

<u>Step 1</u>. Introduce a mathematical model for the Harmonic oscillator.

This is furnished by the theory of Newtonian Mechanics.

- Hooke's Law $\Rightarrow$ the spring exerts a force on the mass $F_{spring}(t) = -k \cdot y(t)$ for some $k > 0$.
- Frictional forces exert a force on M given by $F_{fric}(t) = -b \cdot y'(t)$ for some $b \geq 0$.
- Newton's 2nd Law $\Rightarrow$ the total force exerted on M satisfies $F_{tot}(t) = m \cdot y''(t)$ where m is the mass of M.

Thus Newtonian mechanics implies that $y(t)$ satisfies the <u>differential equation</u> $m \cdot y''(t) = -k \cdot y(t) - b \cdot y'(t)$.

or equivalently $m \cdot y''(t) + b\,y'(t) + k\,y(t) = 0$.

To obtain a unique solution we must impose <u>initial conditions</u>. For example $y(0) = 1$, $y'(0) = 0$.

<u>Step 2</u>. Analyze / solve the initial value problem (IVP)

$$m \cdot y'' + by' + ky = 0 \quad \text{with } y(0) = 1, \; y'(0) = 0.$$

<u>Question</u>: Based on our physical intuition, can we predict what kind of functions will satisfy the IVP above?

<u>Case 1</u>. $b = 0$. (Simple Harmonic Oscillator).

M will oscillate back and forth with a fixed period and amplitude forever. In fact

$$y(t) = \cos\left(\sqrt{\frac{k}{m}}\, t\right)$$ is an explicit

solution to the IVP.

Case 2. $b \neq 0$. We trial the solution

$$y(t) = e^{r \cdot t}$$

where r is a fixed constant.

Then $m \cdot y'' + b \cdot y' + k \cdot y = e^{r \cdot t}(m \cdot r^2 + b \cdot r + k)$.

Hence $y(t) = e^{r \cdot t}$ satisfies the ODE if and only if $m \cdot r^2 + b \cdot r + k = 0$.

So $r = \dfrac{-b \pm \sqrt{b^2 - 4mk}}{2m}$. We naturally obtain three subcases.

Case 2.1. $b^2 - 4mk < 0$ [two complex conjugate roots]

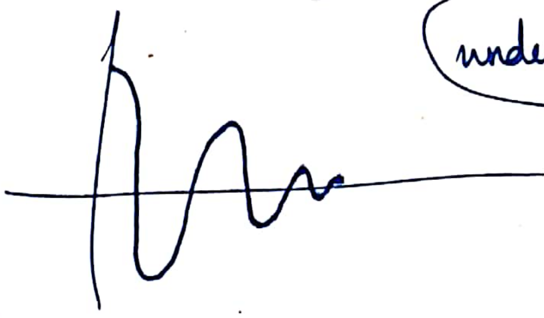
Case 2.2. $b^2 - 4mk = 0$. [one real double root]

Case 2.3. $b^2 - 4mk > 0$. [two real roots]

Case 2.1. ($b^2 - 4mk < 0$).

In this case frictional forces are small compared to $4mk$. Hence one expects the system to oscillate with the amplitude of oscillation decaying. This is termed underdamped oscillation since the damping effect of friction is not sufficiently large to prevent oscillations.

underdamped.



Case 2.2. ($b^2 - 4mk = 0$).

This is the case of critically damped oscillation. The effect of friction is just barely strong enough to prevent oscillation. The spring returns to it's equilibrium exponentially fast.

Case 2.3. ($b^2 - 4mk > 0$). This is the case of overdamped oscillation. Qualitatively this case is similar to Case 2.2. However the spring returns to equilibrium more slowly.